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FINITE DEFLECTION OF A CIRCULAR VISCOPLASTIC PLATE SUBJECT TO PROJECTILE IMPACT

TOMASZ WIERZBICKI

Institute for Basic Technical Research, Polish Academy of Sciences, Warsaw, Poland

and

JAMES M. KELLY

Division of Structural Engineering and Structural Mechanics, University of California, Berkeley, California

Abstract—A method is presented for the determination of the permanent deflection of plates subject to the impact of projectiles moving with velocities high enough to produce plastic deformations in association with finite deflections. The plate material is assumed to obey the Mises–Huber yield condition and its associated flow rule for static deformations and to behave as a viscoplastic solid for dynamic deformations, elastic strains being considered negligible. The method is illustrated by application to the case of a clamped plate struck at the center by a projectile of negligible radius. The relevance of the solution to experimental studies of the dynamic plastic behavior of plates is discussed.

1. INTRODUCTION

THE problem of deformation of thin metal plates impacted by a projectile moving with a velocity sufficiently high to produce an appreciable permanent deflection is of considerable interest from the point of view of both technical application and the possibilities of extensive laboratory experimentation. While the former area of application is obvious, the latter may provide a powerful method for the comparison of various theoretical predictions and the subsequent verification of constitutive equations formulated for combined stresses. Tests of the kind discussed here are characterized by a relative ease of performance and interpretation.

Hitherto the problem of the motion of a plate due to the impact of a moving mass received little attention, mainly because of difficulties involved in the solution of the mathematical problem. The properly posed boundary value problem requires consideration of an interaction between mass and plate without assuming *a priori* the history of the contact force acting on the plate. Even in the elastic range the theoretical investigations are very fragmentary and incomplete [4, 5, 11].

Consider the following physical situation. A rigid–viscoplastic circular plate of thickness $2h$ and outer radius R which is clamped along its circumference is struck centrally by a rigid mass M of negligible radius moving with velocity V. Initially there is a discontinuous velocity profile with the plate center moving at the prescribed velocity V while the remainder of the plate is at rest. Since the speed of shear and bending waves for a rigid–viscoplastic material is infinite any disturbance applied at the plate center $r = 0$ will be felt instantaneously over the entire plate. As time proceeds the discontinuous velocity profile is smoothed out. Each particle of the plate (except $r = 0$) is first accelerated and then decelerated while the velocity of the projectile is diminishing and at a certain instant the entire plate is brought to rest. In the transient problem there is an interaction between the projectile and plate in which the kinetic energy of the moving mass is dissipated into plastic work. It was shown for impulsively loaded beams by Bodner and Symonds [13] that if the ratio of the kinetic energy input to the maximum elastic energy is at least three then neglecting the elastic components of the strain by assuming a rigid–plastic response is permissible. Florence [3] has shown that this result is equally applicable to impulsively loaded plates.

For large values of the impacting energy large deflections may be produced which are accompanied by the elongation of the plate middle surface and development of membrane forces. The interaction of bending moments and membrane forces and the changes in the plate geometry result in a decrease in the permanent plate deflection. A study of this problem for rigid–perfectly plastic material and Tresca yield condition has recently been presented by Jones [6, 7] under the assumption that the plate is simply supported and loaded either impulsively or by a rectangular pressure pulse. Comparison of Jones' theoretical solution with Florence's experimental data [3] exhibited good agreement for aluminum plates but systematic deviations for mild steel plates which may be attributed to the strain rate effect experimentally confirmed for ductile metals with a sharp yield point.

An attempt is made in the present paper to include the strain rate effect into the analysis of large deflections of clamped circular plates subjected to the impact of a rigid mass. The present paper is based to a large extent on the authors' previous paper [8] concerning the bending solution for a similar problem. An interesting conclusion drawn in that paper was that the final central deflection of the plate was little affected by the inertia of the plate itself and was equal in the first approximation to the deflection of a massless plate. The dynamic response of such a plate can thus be analysed by considering the deflection process of the plate as being quasi-static. The static problem equivalent to the dynamical one is that of a concentrated point force. Therefore reference is made here to theoretical solutions and experimental investigations of the load-carrying capacity of plastic plates at moderately large deflection [9, 10]. These results are used to compute the final plate deflection of the present dynamical problem.

2. BENDING SOLUTION FOR SMALL DEFLECTION

The impact of a projectile on a viscoplastic plate was analyzed in [8] under certain simplifying assumptions concerning the geometry of the mass–plate system, properties of the plate material and conditions of loading. The striking mass in the form of a cylinder with negligible radius was considered to be entirely rigid. The plate material was assumed to be rigid–viscoplastic and the constitutive equations employed were those of Hohenemser and Prager with the Mises–Huber yield condition. Only small deflections and bending action of the plate were considered and in the equation of motion the transverse inertia term was retained. To linearize the non-linear constitutive equations a hypothesis of proportional loading was introduced, an idea which in a previous problem of dynamics of circular plates [12] led to results which were in general agreement with both experimental data and theoretical solutions derived from the exact equations. The problem was solved including the time variable boundary condition at the centre of the plate where the deceleration of the impacting mass was proportional to the shearing force Q at $r = 0$. Within the above assumptions an exact solution was obtained for the plate deflection as an infinite series of Bessel and exponential functions. This series is slowly convergent in the first, transient stage of motion when all plate points are accelerated but is rapidly convergent in the second stage of motion. A brief summary of the relevent portions analysis is given in the appendix to which reference may be made for definition of the quantities which may appear in the subsequent argument.

An approximate solution was also presented in [8], valid for a certain range of the parameter $\beta = M/\mu R^2$. The simplifications of the exact solution were based on the following arguments.

If the duration of the first stage of motion is very short so that no dissipation of energy occurs, the energy conservation equation allows the determination of the initial velocity $V_0 f(r)$ of the mass–plate system for a certain velocity profile of the plate

$$\tfrac{1}{2}M(V^2 - V_0^2) = V_0^2 2\pi\mu \int_0^R [f(r)]^2 r \, \mathrm{d}r. \tag{2.1}$$

Equation (2.1) postulates an instantaneous transition of a part of the initial kinetic energy from the projectile to the plate. Furthermore, in the second stage of motion only the first term in the exact solution is significant and the first eigenfunction can be approximated by the velocity profile of the corresponding static problem. The equivalent static problem is a concentrated force acting at the centre of a clamped circular plate. The relevant velocity profile associated with the static load-carrying capacity

$$P_0 = \frac{4\pi M_0}{\sqrt{3}} \tag{2.2}$$

for the Mises–Huber yield condition is

$$f(r) = 1 - (r/R)^2[1 - 2\log_e(r/R)] \tag{2.3}$$

where $M_0 = \sigma_0 h^2$ is the fully plastic yield moment. Introducting equation (2.3) into equation (2.1) we find the relation between initial mass velocity V and the central velocity of the plate V_0

$$V_0^2 = \frac{V^2}{(1 + 4/10\beta)}. \tag{2.4}$$

Thus the value of the permanent deflection of the plate within the introduced simplification is expressed as,

$$W(r) = \frac{1}{2}MV^2\frac{\varphi(\beta)}{P_0}\left[2\eta - 2\eta^2\log_e\left(1 + \frac{1}{\eta}\right)\right]f(r) \tag{2.5}$$

where

$$\varphi(\beta) = \frac{16\pi}{\beta\lambda_1^4}\frac{1}{1 + 4/10\beta} \tag{2.6}$$

and

$$\eta = \frac{3R^2\gamma}{16hV}\sqrt{(1 + 4/10\beta)} \tag{2.7}$$

The deflection $W(r)$ is equal to the kinetic energy of the projectile multiplied by terms depending on static load-carrying capacity P_0, mass ratio β, viscosity constant γ and dimensionless plate radius r/R.

The first eigenvalue λ_1, appearing in equation (2.6) is a solution of a transcendental equation involving Bessel functions and goes to zero with $\beta \to \infty$. An expansion of the eigenvalue equation for large β, taken to the fifth power yields

$$\lambda_1^4 = \frac{16\pi}{\beta}\left(1+\frac{\pi}{6\beta}\right). \tag{2.8}$$

Now the term responsible for the variation of $W(r)$ with mass ratio is reduced to

$$\varphi(\beta) = \frac{1}{1+\pi/6\beta}\cdot\frac{1}{1+4/10\beta}. \tag{2.9}$$

For meaningful values of β i.e. for $\beta > 1$ the term $\varphi(\beta)$ is equal approximately to unity. If we multiply both sides of equation (2.5) by P_0 we can write the central deflection of the plate $W(0) = \delta$ in the form

$$P_0\delta = \tfrac{1}{2}MV^2[2\eta - 2\eta^2 \log_e(1+1/\eta)]. \tag{2.10}$$

Equation (2.10) gives the permanent deflection of the rigid–viscoplastic plate impacted by heavy projectile for which $\beta > 1$ and the function $\varphi(\beta)$ can be set equal to unity.

In the limiting case when $\gamma \to \infty$ and consequently $\eta \to \infty$ which corresponds to the rigid-perfectly plastic behavior, the term in parenthesis is equal to unity, conversely for $\gamma = 0$ the material is entirely rigid and the plate deflection is zero. The left hand side of equation (2.10) is the dissipation of energy of the equivalent static problem for perfectly plastic material. The right hand side of the same equation represents the kinetic energy input diminished by the term responsible for viscous effects. The conclusion that the plate response in the case of a projectile impact for large β is not significantly influenced by the inertial characteristics of the plate will be essential for the derivation of the solution of a similar problem at large deflection.

3. SOLUTION FOR LARGE DEFLECTION

Extensive investigations on quasi-static flow of rigid–perfectly plastic plates and shells at large deflection have demonstrated that these structures may carry loads much higher than the prediction based on purely bending theory for infinitesimal strains [1, 2, 9, 10]. It was found that appreciable membrane forces are developed at early stages of the deformation process. For some structures such as clamped circular plates or cylindrical shells with axial constraints the stress distribution approaches a membrane state for deflection of the order of the thickness of the relevant structure [1, 9]. Impact of projectiles on metal plates may produce deflections of the order of several plate thicknesses. Therefore it is essential to account for the changes in geometry and effects of membrane forces for a more realistic assessment of the shape and permanent deflection of the plate. The necessity of considering this effect was pointed out by Florence [3], in experimental work concerning a circular plate loaded by a uniformly distributed impulse of pressure.

An approximate method for the estimation of the central deflection of a viscoplastic plate will now be presented. This method is based upon the solution detailed in the previous

Section. Consequently all assumptions concerning the bending solution for small deflections will apply also to the present case and the same refers to the range of validity of the solution to be found. This is true except for the hypothesis of proportional loading which in the case when membrane forces come into play requires modification to be consistent with the Love–Kirchhoff assumption concerning straight normals and plane sections and conditions of axial symmetry. Now the stress trajectory in the four-dimensional space of bending moments and membrane forces is no longer a straight line passing through the origin. Proportional loading would demand straight line trajectories in two subspaces of bending moments and membrane forces separately. The concept of proportional loading used in Section 2 has merely served to linearize the non-linear problem and obtain an explicit formula for the plate deflection. The analysis of this solution has led to the conclusion that for large β the plate can be considered as being massless. However, this property is not a consequence of the hypothesis concerning proportional loading but is rather a general property of the mass–plate interaction.

If we extend now the above mentioned conclusion to the case of moderately large deflections then by analogy with equation (2.10) the final plate deflection can be computed from the equation.

$$\int_0^{\delta} P(\delta)\,\mathrm{d}\delta = \tfrac{1}{2}MV^2[2\eta-\eta^2\log_e(1+1/\eta)]. \tag{3.1}$$

As in the previous case the left hand side of equation (3.1) represents the dissipation of energy of the equivalent static problem but for a variable force $P(\delta)$. The load-carrying capacity of the plate is no longer constant but is uniquely related to the plate deflection δ. From both experimental investigations and theoretical considerations the following formula is valid for clamped circular plates loaded by a point force

$$P(\delta) = P_0[1+\alpha(\delta/h)^2]. \tag{3.2}$$

In fact Onat and Haythornthwaite [10] and Lepik [9] have derived solutions in which the analytic form of the $P = P(\delta)$ dependence was different for different ranges of the deflection. For the sake of convenience one analytic expression is assumed here to be valid throughout the whole deformation process. Formula (3.2) fits the experimental curves presented in [10] for different ratios h/R. Based on these results the parameter α was found to be

R/h	10	20	40
α	0·7	0·45	0·3

The advantage of using equation (3.2) stems also from the fact that static tests were performed using circular punches designed to approximate the point load. In experiments on plates subjected to the impact of a mass the projectiles are also of finite radii the correlation between the values observed experimentally and computed from equation (3.1) would be considerably improved.

Substitution of equation (3.2) into equation (3.1) and integration yields the sought expression for the permanent plate deflection in terms of the kinetic energy of the projectile and strain rate characteristics of the plate material,

$$P_0\left[\frac{\delta}{h}+\frac{\alpha}{3}\left(\frac{\delta}{h}\right)^3\right] = \frac{1}{2}MV^2\left[2\eta-2\eta^2\log_e\left(1+\frac{1}{\eta}\right)\right]. \tag{3.3}$$

4. NUMERICAL EXAMPLE

As an illustration of the solution obtained consider a plate made of C.R. Steel 1018, characterized by the following mechanical parameters:

$$\text{Yield stress in tension } \sigma_0 = 7{\cdot}9 \times 10^4 \text{ lb/in}^2$$
$$\text{Mass density } \rho = 7{\cdot}32 \times 10^{-4} \text{lb sec}^2/\text{in}^4.$$

Taking into account that

$$P_0 = \frac{4\pi M_0}{\sqrt{3}}, \qquad M = \beta\mu R^2, \qquad \mu = 2h\rho, \qquad M_0 = \sigma_0 h^2 \tag{4.1}$$

equation (3.3) in the case $\gamma \to \infty$ yields

$$V^2\left[\frac{\sqrt{3}}{4\pi}\cdot\frac{\rho}{\sigma_0}\cdot\left(\frac{R}{h}\right)^2\beta\right] = \frac{\delta}{h}+\frac{\alpha}{3}\left(\frac{\delta}{h}\right)^3. \tag{4.2}$$

Assume now $\beta = 1$ and $R/h = 20$. The corresponding value of the parameter α is $\alpha = 0{\cdot}45$. Substituting these values into equation (4.2) we finally reach

$$V/10^3 = 0{\cdot}116\sqrt{\left[\frac{\delta}{h}+0{\cdot}15\left(\frac{\delta}{h}\right)^3\right]}$$

where units of the projectile velocity V are in ft/sec.

The plot of the central plate deflection versus the projectile velocity for $\gamma \to \infty$ is presented in Fig. 1, full line. The broken line on the same figure denotes the purely bending solution computed from formula (2.10). Consideration of the viscous term

$$2\eta - 2\eta^2 \log_e(1+1/\eta)$$

equation (3.3), diminishes the permanent plate deflection depending upon the choice of the viscosity constant. This is shown on Fig. 1 for a few chosen values of the viscosity constant.

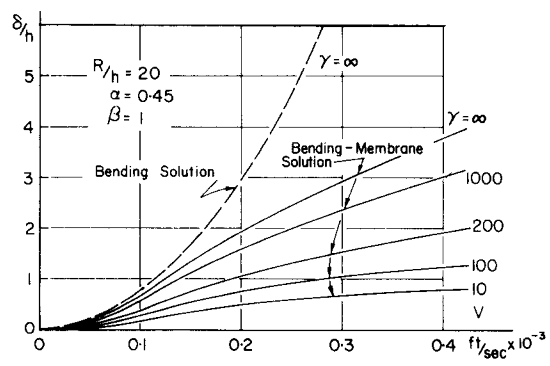

FIG. 1. Permanent central deflection as a function of velocity for various values of viscosity constant.

For the strain rates encountered in a plate impact test the values of γ for different mild steels may be within the range between $\gamma = 200 \text{ sec}^{-1}$ and 1000 sec^{-1}. An appreciable deviation between the bending and bending-membrane solutions is observed for the deflections in the range.

5. CONCLUDING REMARKS

Many of the experimental studies of the dynamic plastic behavior of plates have given results which are quite different from the predictions of the rate independent bending theory based on the Tresca yield condition. The Tresca yield condition is in a sense a linearization of a more exact yield condition and the flow rule associated with the piecewise linear yield condition requires velocity fields which have piecewise constant strain rate vectors. Such velocity fields are clearly unrealistic but probably are not the cause of the discrepancies between theory and experiment. The extent to which the experimental deviations are the result of membrane forces or derive from strain rate effects is difficult to estimate in the absence of a theoretical solution which incorporates both of these effects. It was noted for example in [12] that the inclusion of rate effects, and the use of a Mises–Huber yield condition could provide correlation with experimental data which had previously been thought to be only the result of membrane effects.

The experimental situation which corresponds to the solution presented here is comparatively easily developed requiring only an accurate measurement of the projectile velocity and the plate deflection. The physical constants involved in the theory which are the static yield stress and viscoplastic constant γ can be obtained by independent tests. Thus the solution may be used to study the relative importance of rate effects and membrane forces in the plate behavior and to determine to what extent one or the other may be neglected in the analysis of plates of various materials and dimensions. The result given for the clamped plate can be extended with some increase in the algebraic complexity of the solution to other support conditions.

It is to be expected that the rate effects would be of considerable importance in plates of mild steel which is known to be extremely rate sensitive but some aluminum alloys particularly 1060 are also susceptible to this influence.

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APPENDIX

A brief summary of the analysis on which the development of Section 2 is based is given here.

The notation used is as follows: all quantities are assumed to be functions only of r, the distance measured from the plate center, and of time, t. The surface tractions applied to the plate are positive in the direction of positive transverse displacements of the middle surface. The velocity of points of the middle surface is v. The transverse shear force is denoted by Q and the radial and circumferential bending moments are M_r and M_φ respectively. The plate radius is R, the thickness is 2 h, and the mass density per unit area of the plate middle surface is μ. The equations of motion are

$$(rQ)_{,r}+rp = -\mu r v_{,t} \tag{A1}$$

$$(rM_r)_{,r}-M_\varphi = rQ, \tag{A2}$$

and the kinematics of the deformation require that the rates of curvature K_r and K_φ be related to the velocity v through

$$K_r = -v_{,rr}, \qquad K_\varphi = -\frac{1}{r}v_{,r}. \tag{A3}$$

The stress strain rate relations of the plate material are expressed analytically as

$$\varepsilon_{ij} = \gamma\frac{(S_{kl}S_{kl}/2)^{\frac{1}{2}}-k}{(S_{kl}S_{kl}/2)^{\frac{1}{2}}}\cdot\frac{S_{ij}}{k} \tag{A4}$$

for $\frac{1}{2}S_{ij}S_{ij} \geq k^2$, where ε_{ij} is the strain rate tensor and S_{ij} the stress deviator. The quantities k and γ are material constants, k being the static yield stress in simple shear and $1/\gamma$ a natural time. For a thin plate with axial symmetry the above constitutive relations in terms of moments and curvature rates take the form

$$\left.\begin{aligned} K_r &= B\left(1-\frac{M_0}{\sqrt{(M_r^2-M_rM_\varphi+M_\varphi^2)}}\right)\frac{2M_r-M_\varphi}{M_0}, \\ K_\varphi &= B\left(1-\frac{M_0}{\sqrt{(M_r^2-M_rM_\varphi+M_\varphi^2)}}\right)\frac{2M_\varphi-M_r}{M_0}, \end{aligned}\right\} \tag{A5}$$

where

$$B = \frac{\sqrt{3}\gamma}{2h} \quad \text{and} \quad M_0 = \sqrt{3}kh^2.$$

The plate is assumed to be clamped at the outer edge and the radius of the impacting mass to be negligible in comparison to R. As a result of these assumptions the boundary conditions are

$$\text{at } r = R, \qquad v = 0, \qquad v_{,r} = 0 \tag{A6}$$

and

$$\text{at } r = 0, \qquad \lim_{r\to 0} v < \infty, \qquad \lim_{r\to 0} 2\pi rQ = Mv_{,t}|_{r=0} \tag{A7}$$

where M is the mass of the projectile and the initial conditions are

$$\text{at } t = 0 \qquad v = 0 \text{ for } r \neq 0, \qquad v = V \text{ for } r = 0 \tag{A8}$$

where V is the impacting velocity of the projectile.

The system of equations (A1) to (A5) is a complete one but due to the nonlinear form of (A5) cannot be treated analytically unless the moment curvature relationships can be linearized. This is achieved by a method which depends on assuming, in the nine dimensional space of the stress deviator, that the stress trajectory for any particle of the material is a straight line. Thus the quantity $S_{ij}/(\frac{1}{2}S_{kl}S_{kl})^{\frac{1}{2}}$ = const. and the stress strain relation becomes

$$\varepsilon_{ij} = \frac{1}{\tau}(S_{ij} - \bar{S}_{ij}), \tag{A9}$$

where $\bar{S}_{ij}$ is the state of stress on the surface $\frac{1}{2}S_{kl}S_{kl} = k^2$. The corresponding equations relating moments and curvature rates are

$$\begin{aligned} K_r &= \frac{B}{M_0}[(2M_r - M_\varphi) - (2\bar{M}_r - \bar{M}_\varphi)], \\ K_\varphi &= \frac{B}{M_0}[(2M_\varphi - M_r) - (2\bar{M}_\varphi - \bar{M}_r)], \end{aligned} \tag{A10}$$

where $\bar{M}_r$ and $\bar{M}_\varphi$ are moments satisfying the initial yield condition

$$M_r^2 - M_r M_\varphi + M_\varphi^2 = M_0^2.$$

The equations of motion, kinematics and the reduced moment curvature relation equations (10) taken together lead to the dimensionless equations

$$\Delta^4 u + \alpha u_{,t} = 0, \qquad \Delta^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \tag{A11}$$

where

$$u = \frac{v}{BR^2}, \qquad \rho = \frac{r}{R}, \qquad \alpha = \frac{3}{2}\frac{BR^4}{M_0}\mu, \qquad \beta = \frac{M}{\mu R^2},$$

and the boundary condition (7) becomes

$$\lim_{\rho \to 0} 2\pi\rho(\Delta^2 u)_{,\rho} = -2\pi\sqrt{3} - \alpha\beta u_{,t}|_{\rho=0}. \tag{A12}$$

The solution of the above equation may be expressed as

$$u(\rho, t) = \sum_{n=1}^{\infty} A_n \psi_n(\rho)\, e^{-(\lambda_n^4/\alpha)t} - f_0(\rho), \tag{A13}$$

where $\psi_n(\rho)$ is the solution of the following equation

$$\Delta^4 \psi_n + \lambda_n^4 \psi^4 = 0, \tag{A14}$$

with

$$\psi_n(1) = 0, \qquad \psi_{n,\rho}(1) = 0, \qquad \lim_{\rho \to 0} \psi_n < \infty, \qquad \lim_{\rho \to 0} 2\pi\rho(\Delta^2 \psi_n)_{,\rho} = \beta\lambda_n^4 \psi_n^4(0), \tag{A15}$$

and $f_0(\rho)$ satisfies

$$\Delta^4 f_0 = 0, \tag{A16}$$

with

$$f_0(1) = 0, \qquad f_{0,\rho}(1) = 0, \qquad \lim_{\rho \to 0} f_0 < \infty, \qquad \lim_{\rho \to 0} 2\pi\rho(\Delta^2 f_0)_{,\rho} = 2\pi\sqrt{3}. \tag{A17}$$

The equation for the eigenvalues λ_n is obtained from the boundary conditions (15) as

$$\begin{gathered} J_0(\lambda_n)I_1(\lambda_n) + J_1(\lambda_n)I_0(\lambda_n) - \frac{\lambda_n^2\beta}{8}\left\{\left[Y_0(\lambda_n) + \frac{2}{\pi}K_0(\lambda_n)\right][J_1(\lambda_n) + I_1(\lambda_n)]\right. \\ \left. -\left[Y_1(\lambda_n) + \frac{2}{\pi}K_1(\lambda_n)\right][J_0(\lambda_n) - I_0(\lambda_n)]\right\} = 0, \end{gathered} \tag{A18}$$

and the resulting eigenfunctions $\psi_n(\rho)$ are

$$\psi_n(\rho) = \frac{1}{2}[J_0(\lambda_n\rho) + I_0(\lambda_n\rho)] - \frac{\lambda_n^2\beta}{8}\left[Y_0(\lambda_n\rho) + \frac{2}{\pi}K_0(\lambda_n\rho)\right] - \frac{\alpha_n}{2}[J_0(\lambda_n\rho) - I_0(\lambda_n\rho)] \tag{A19}$$

where

$$\alpha_n = \frac{J_0(\lambda_n) + I_0(\lambda_n) - \lambda_n^2\beta/4[Y_0(\lambda_n) + 2/\pi K_0(\lambda_n)]}{J_0(\lambda_n) - I_0(\lambda_n)}. \tag{A20}$$

The eigenfunctions $\psi_n(\rho)$ are not orthogonal in the usual sense but can be considered to be orthogonal with respect to the weight function

$$\rho[1 + \beta\delta(\rho)]$$

where $\delta(\rho)$, the Dirac delta function, is defined by

$$\delta(\rho) = 0 \quad \text{for} \quad \rho \neq 0,$$

and

$$2\pi\int_0^1 \rho\delta(\rho)\,\mathrm{d}\rho = 1.$$

We define

$$(\psi_n, \psi_m) = \int_0^1 \rho[1 + \beta\delta(\rho)]\psi_n(\rho)\psi_m(\rho)\,\mathrm{d}\rho,$$

and

$$|\psi_n| = (\psi_n, \psi_n)^{\frac{1}{2}}.$$

From the initial condition (8) we obtain

$$A_n|\psi_n|^2 - (f_0, \psi_n) = \beta V\psi_n(0)/BR^2$$

leading to

$$u(\rho, t) = \frac{\beta V}{BR^2}\sum_{n=1}^{\infty}\psi_n(\rho)\left\{\frac{1}{|\psi_n|^2}\mathrm{e}^{-(\lambda_n^4/\alpha)t} - \frac{(f_0, \psi_n)}{|\psi_n|^2}[1 - \mathrm{e}^{-(\lambda_n^4/\alpha)t}]\right\}. \tag{A21}$$

Deflection of the plate can be obtained through the time-wise integration of the latter equation.

A good approximation to the solution can be obtained by using only the first term in the expansion and by replacing $\psi(\rho)$ by $f_0(\rho)/f_0(0)$. The solution then takes the form

$$u(\rho, t) = Af_0(\rho)\,e^{-(\lambda_1^4/\alpha)t} - f_0(\rho) \tag{A22}$$

This is true except at the beginning of the motion. At the instant of impact the velocity satisfies the initial condition (8) but immediately after there is a sudden transition from the situation when only the center of the plate is moving with prescribed velocity, V, while the remainder of the plate is at rest, to the state when the velocity field is given by the function $V_0 f_0(\rho)/f_0(0)$. At this time the velocity of the center of the plate is V_0 less than V.

The possibility of a sudden change in velocity is acceptable within the assumed model of rigid viscoplastic material since the wave speed approaches infinity and the disturbance at the plate center is instantaneously transmitted to arbitrary points of the plate.

We assume that the process takes place in such a short time that no loss of energy occurs, thus conservation of energy allows the computation of the appropriate initial central velocity. The energy balance equation is

$$\tfrac{1}{2}M(V^2 - V_0^2) = 2\pi R^2 \mu V_0 \int_0^1 [f_0(\rho)/f_0(0)]^2 \rho\, d\rho \tag{A23}$$

from which we obtain

$$V_0^2 = \frac{V^2}{(1+4/10\beta)} \tag{A24}$$

From the initial condition $u(0,0) = V_0/BR^2$ we find

$$A = u_0 + \frac{\sqrt{3}}{8}.$$

the time t_f at which the mass is brought to rest is given by

$$t_f = -\frac{\alpha}{\lambda_1^4}\ln\left(\frac{\sqrt{3/8}}{u_0+\sqrt{3/8}}\right),$$

and the resultant deflection, δ, of the central point is given by

$$\frac{\delta}{R} = \frac{\alpha}{\lambda_1^4}u_0 - \frac{\sqrt{3}}{8}\frac{\alpha}{\lambda_1^4}\ln\left[\frac{\sqrt{3/8}}{u_0+\sqrt{3/8}}\right].$$

In physical quantities the permanent deflection of the plate becomes

$$\delta = \frac{6\mu R^2 V^2}{\sqrt{3}M_0\lambda_1^4}\frac{1}{1+4/10\beta}\left\{2\eta - 2\eta^2 \ln\left(1+\frac{1}{\eta}\right)\right\}, \tag{A25}$$

where

$$\eta = \frac{\sqrt{3}R^2 B}{8V}\bigg/\sqrt{\left(1+\frac{4}{10\beta}\right)}.$$

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Абстракт—Представлен метод для определения остаточного прогиба пластинок, подверженных удару снарядов, движущихся со скоростями достаточуо большими для того, чтобы вызвать пластические деформации и конечные прогибы. Подразумивается, что материал удовлетворяет условию текучести Мизеса-Губера и ассоцированному закону течения для статических деформаций, и ведет себя как вязкопластический материал для динамических деформаций, причем упругие деформации пренебрегаемо малы. Метод иллюстрируется на задаче о защемленной пластинке, подвергающейся центральному удару снарядом небольшого радиуса. Рассматриваются соответствие решения с экспериментальными решениями динамического пластического поведения пластинок.